L-functions and Class Numbers

Student Number Theory Seminar

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We follow Romyar Sharifi's *Notes on Iwasawa Theory*, with some help from Neukirch's *Algebraic Number Theory*.

1 L-functions of Dirichlet Characters

1.1 Dirichlet Characters

A *Dirichlet character* is a completely multiplicative function $\chi : \mathbb{Z} \to \mathbb{C}$ which is periodic with some period n and satisfies $\chi(a) \neq 0$ precisely when $(a, n) = 1$. A Dirichlet character can also be regarded a character $\chi : (\mathbb{Z}/n\mathbb{Z})^* \to \mathbb{C}$. It is not terribly hard to see these two notions are equivalent, and we will use this equivalence without warning (though we will use the second notion whenever possible).

The *conductor* f_χ of a Dirichlet character $\chi : (\mathbb{Z}/n\mathbb{Z})^* \to \mathbb{C}$ is the smallest positive integer f such that χ factors through $(\mathbb{Z}/f\mathbb{Z})^*$.

$$\begin{tikzcd}
(\mathbb{Z}/n\mathbb{Z})^* \arrow[rr, "\chi"] \arrow[dr, "\bmod f"'] & & \mathbb{C} \\
& (\mathbb{Z}/f\mathbb{Z})^* \arrow[ur, "\tilde{\chi}"'] &
\end{tikzcd}$$

We say that a Dirichlet character is *primitive* if its conductor is equal to its period (i.e. it does not factor as above). From any Dirichlet character $\chi : (\mathbb{Z}/n\mathbb{Z})^* \to \mathbb{C}$ of conductor f we can produce an associated primitive Dirichlet character of period and conductor f by taking the character $\tilde{\chi} : (\mathbb{Z}/f\mathbb{Z})^* \to \mathbb{C}$ in the above diagram. If we regard Dirichlet characters as maps $\mathbb{Z} \to \mathbb{C}$, then $\tilde{\chi}$ is the Dirichlet character with least period such that $\tilde{\chi}(a) = \chi(a)$ whenever $(a, n) = 1$. (Intuitively, we're "filling in" as many zeros as possible in the non-primitive Dirichlet character).

We say a Dirichlet character χ is even or odd according as $\chi(-1) = 1$ or $\chi(-1) = -1$, respectively.

Table 1: A Dirichlet character of period 14 and its associated primitive character of period 7. Here $\zeta = e^{2\pi i/6}$ is a 6^{th} root of unity.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	period	conductor	parity
χ	0	1	0	ζ	0	ζ^5	0	0	0	ζ^2	0	ζ^4	0	-1	14	7	odd
$\tilde{\chi}$	0	1	ζ^2	ζ	ζ^4	ζ^5	-1	0	1	ζ^2	ζ	ζ^4	ζ^5	-1	7	7	odd

1.2 L-functions

To every Dirichlet character χ we associate an L-series $L(\chi, s)$, defined by

$$L(\chi, s) = \sum_{n \geq 1} \frac{\chi(n)}{n^s}.$$

This series converges absolutely for $s \in \mathbb{C}$ with $\mathrm{re}(s) > 1$, and converges uniformly on $\mathrm{re}(s) > 1 + \varepsilon$ (for any $\varepsilon > 0$). The complete multiplicativity of Dirichlet characters imply that the L-series has an Euler product:

$$L(\chi, s) = \prod_{p \text{ prime}} \frac{1}{1 - \chi(p)p^{-s}}$$

for $\mathrm{re}(s) > 1$.

Theorem 1.1. *The L-series $L(\chi, s)$ has a meromorphic continuation to the whole complex plane (which we also denote $L(\chi, s)$). If χ is not the trivial character then $L(\chi, s)$ is in fact holomorphic, while if χ is trivial then $\zeta(s) = L(\chi, s)$ has a simple pole with residue* 1 *at* $s = 1$.

This meromorphic continuation we call the *L-function* of χ.

To express the special values of L-functions and their functional equations, we'll need Gauss sums. The *Gauss sum* associated to a Dirichlet character χ of period n is

$$\tau(\chi) = \sum_{a=1}^{n} \chi(a) e^{2\pi i a/n}.$$

The next lemma records some of their basic properties.

Lemma 1.2. *If χ is a primitive Dirichlet character then*

$$|\tau(\chi)| = \sqrt{f_\chi},$$

and for all $b \in \mathbb{Z}$,

$$\chi(b)\tau(\overline{\chi}) = \sum_{a=1}^{f_\chi} \overline{\chi}(a) e^{2\pi i ab/f_\chi}.$$

(Here $\overline{\chi}(a) = \overline{\chi(a)}$ is the conjugate Dirichlet character).

In order to give a functional equation for our L-functions, we make the following defintions. Let

$$\delta_\chi = \frac{1 - \chi(-1)}{2} = \begin{cases} 0 & \text{if } \chi \text{ is even} \\ 1 & \text{if } \chi \text{ is odd} \end{cases},$$

an indicator of whether χ is even or odd; let

$$\varepsilon_\chi = \frac{\tau(\chi)}{i^{\delta_\chi}\sqrt{f_\chi}},$$

some algebraic number (of absolute value 1); and let

$$\Lambda(\chi, s) = \left(\tfrac{f_\chi}{\pi}\right)^{s/2} \Gamma\left(\tfrac{s+\delta_\chi}{2}\right) L(\chi, s).$$

Recall that $L(\chi, s)$ is a product over primes; we should think of the Γ-function in the above expression as adding in the infinite prime.

We have the following functional equation for Λ.

Theorem 1.3. *For χ a primitive Dirichlet character,*

$$\Lambda(\chi, s) = \varepsilon_\chi \Lambda(\overline{\chi}, 1 - s).$$

1.3 Special Values

We are interested in these L-functions for their special values (at integers). In order to compute these special values we introduce Bernoulli numbers, along with a slight generalization.

Define the Bernoulli numbers B_n for $n \geq 0$ by

$$\frac{t}{e^t - 1} = \sum_{n \geq 0} B_n \frac{t^n}{n!}.$$

The inverse of this power series is

$$\frac{e^t - 1}{t} = \sum_{n \geq 0} \frac{t^n}{(n+1)!},$$

and we can use this fact to inductively compute the B_n.

Table 2: The first few Bernoulli numbers.

n	0	1	2	3	4	5	6	7	8	9	10
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$

Note in particular that $B_n = 0$ for all odd $n > 1$ (which can be seen by showing that $\frac{t}{e^t-1} + \frac{1}{2}t$ is an even function).

For χ a primitive Dirichlet character, define generalized Bernoulli numbers $B_{n,\chi}$ for $n \geq 0$ by

$$\sum_{a=1}^{f_\chi} \chi(a) \frac{te^{at}}{e^{f_\chi t} - 1} = \sum_{n \geq 0} B_{n,\chi} \frac{t^n}{n!}.$$

In fact in this definition we can replace f_χ by any multiple of it, using the identity

$$\sum_{k=0}^{r-1} \frac{x^k}{x^r - 1} = \frac{1}{x - 1}.$$

Note that $\tilde{\chi}$ is odd, and $B_{n,\tilde{\chi}} = 0$ for even n. In general, $B_{n,\chi} = 0$ for $n \not\equiv \delta_\chi \bmod 2$, with the single exception of $B_{1,1} = \frac{1}{2}$ (where 1 denotes the trivial character).

Now we can give the following special values for our L-functions.

Table 3: The first few Bernoulli numbers associated to the primitive character $\tilde{\chi}$ of Table 1.1.

n	0	1	2	3	4	5	6	7	8	9	10
$B_{n,\tilde{\chi}}$	0	$-\frac{4}{7}-\frac{2\sqrt{3}}{7}i$	0	$3+3\sqrt{3}i$	0	$-\frac{445}{7}-\frac{565\sqrt{3}}{7}i$	0	$\frac{22249}{7}+\frac{30049\sqrt{3}}{7}i$	0	$-281223-385551\sqrt{3}i$	0

Proposition 1.4. *Let χ be a primitive Dirichlet character. Then for all integers $n \geq 1$, we have*

$$L(\chi, 1-n) = -\frac{B_{n,\chi}}{n}.$$

Proof. Complex analysis. □

Theorem 1.5. *Let χ be a non-trivial primitive Dirichlet character. Then*

$$L(\chi, 1) = \begin{cases} \dfrac{\pi i \tau(\chi)}{f_\chi} B_{1,\overline{\chi}} & \text{if } \chi \text{ is odd,} \\ -\dfrac{\tau(\chi)}{f_\chi} \displaystyle\sum_{a=1}^{f_\chi} \overline{\chi}(a) \log\left|1 - e^{2\pi i a/f_\chi}\right| & \text{if } \chi \text{ is even.} \end{cases}$$

Proof. Odd case: functional equation and $\Gamma(1/2) = \sqrt{\pi}$ give

$$\begin{aligned} \Lambda(\chi, 1) &= \varepsilon_\chi \Lambda(\overline{\chi}, 0) \\ \left(\tfrac{f_\chi}{\pi}\right)^{1/2} \Gamma\left(\tfrac{1+1}{2}\right) L(\chi, 1) &= \frac{\tau(\chi)}{i^1 \sqrt{f_\chi}} \left(\tfrac{f_\chi}{\pi}\right)^{0/2} \Gamma\left(\tfrac{0+1}{2}\right) L(\overline{\chi}, 0) \\ L(\chi, 1) &= -\frac{\tau(\chi) i \pi}{f_\chi} L(\overline{\chi}, 0) \\ L(\chi, 1) &= \frac{\tau(\chi) i \pi}{f_\chi} B_{1,\overline{\chi}} \end{aligned}$$

Even case: using the Gauss sum formula, the changing the order of summantion, then recognizing the power series of $-\log(1-z)$,

$$\begin{aligned} L(\chi, 1) = \sum_{n\geq 1} \frac{\chi(n)}{n} &= \sum_{n \geq 1} \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{f_\chi} \frac{\overline{\chi}(a) e^{2\pi i a n/f_\chi}}{n} \\ &= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{f_\chi} \overline{\chi}(a) \sum_{n\geq 1} \frac{e^{2\pi i a n/f_\chi}}{n} \\ &= -\frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{f_\chi} \overline{\chi}(a) \log(1 - e^{2\pi i a/f_\chi}) \end{aligned}$$

Since χ is even we have $\tau(\overline{\chi}) = \overline{\tau(\chi)}$, so $f_\chi = \tau(\chi)\tau(\overline{\chi})$. Also since χ is even and we are summing over all $a \bmod f_\chi$, we can replace the log with

$$\frac{1}{2}\left(\log(1 - e^{2\pi i a/f_\chi}) + \log(1 - e^{2\pi i (f_\chi - a)/f_\chi})\right) = \log\left|1 - e^{2\pi i a/f_\chi}\right|.$$

This changes the above equation to

$$L(\chi, 1) = -\frac{\tau(\chi)}{f_\chi} \sum_{a=1}^{f_\chi} \overline{\chi}(a) \log\left|1 - e^{2\pi i a/f_\chi}\right|$$

as desired. □

2 ζ-functions of Number Fields

2.1 ζ-functions and L-functions

Recall the *norm* of an ideal $\mathfrak{a} \subset \mathcal{O}_F$ to be $N(\mathfrak{a}) = [\mathcal{O}_F : \mathfrak{a}]$. We define a ζ-series for a number field F by

$$\zeta_F(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_F} \frac{1}{N(\mathfrak{a})^s},$$

where the sum is over non-zero ideals of $\mathcal{O}_F$. Note that $\zeta_{\mathbb{Q}}(s) = \zeta(s)$ is the classical Riemann ζ-function. As in the case of L-series above, this ζ-series has an Euler product,

$$\zeta_F(s) = \prod_{\mathfrak{p} \subset \mathcal{O}} \frac{1}{1 - N(\mathfrak{p})^{-s}},$$

where the product is over non-zero prime ideals of $\mathcal{O}_K$. These ζ-series also have a meromorphic continuation and functional equation.

Now suppose F is an abelian extension of $\mathbb{Q}$. Then $\mathbb{Q} \subset F \subset \mathbb{Q}(\mu_n)$ for some n, and

$$\mathrm{Gal}(F/\mathbb{Q}) \cong \mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q}) \Big/ \mathrm{Gal}(\mathbb{Q}(\mu_n)/F) \cong (\mathbb{Z}/n\mathbb{Z})^* \Big/ \mathrm{Gal}(\mathbb{Q}(\mu_n)/F)$$

realizes $\mathrm{Gal}(F/\mathbb{Q})$ as a quotient of $(\mathbb{Z}/n\mathbb{Z})^*$. Given a character of $\mathrm{Gal}(F/\mathbb{Q})$, we can lift it to a character of $(\mathbb{Z}/n\mathbb{Z})^*$, which has an associated primitive Dirichlet character. Define $X(F)$ to be the set of Dirichlet characters produced in this way, i.e. the set of primitive Dirichlet characters associated to characters of $(\mathbb{Z}/n\mathbb{Z})^*$ that factor through $\mathrm{Gal}(F/\mathbb{Q})$. We have the following relationship between the ζ-function of F and the L-functions of these Dirichlet characters.

Proposition 2.1. *For F an abelian field,*

$$\zeta_F(s) = \prod_{\chi \in X(F)} L(\chi, s).$$

(Note by examining both sides' poles at $s = 1$, we can see that $L(\chi, 1) \neq 0$ for a non-trivial character χ, and this can be used to prove Dirichlet's theorem on primes in arithmetic progressions.)

Proof. Let p be a prime which decomposes in F as $p = (\mathfrak{p}_1 \cdots \mathfrak{p}_r)^e$, with $N(\mathfrak{p}_i) = f$. Then p contributes $(1 - p^{-fs})^{-r}$ to the Euler product of $\zeta_F(s)$, and contributes $\prod_{\chi \in X(F)}(1 - \chi(p)p^{-s})^{-1}$ to the product of L-functions. We want to show that these are the same. Seems like this is done essentially by carefully examining how primes split, but the rest of the proof isn't clear to me. $\square$

For example, consider $F = \mathbb{Q}(i)$. Then $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q}) \cong (\mathbb{Z}/4\mathbb{Z})^*$, which has two associated primitive Dirichlet characters: the trivial character 1, and the character χ defined by

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \bmod 4, \\ -1 & \text{if } n \equiv 3 \bmod 4. \end{cases}$$

In this field 2 ramifies, primes $p \equiv 1 \bmod 4$ split completely, and primes $p \equiv 3 \bmod 4$ are inert.

Thus

$$\begin{aligned}
\zeta_{\mathbb{Q}(i)}(s) &= (1-2^{-s})^{-1} \prod_{p\equiv 1 \bmod 4} (1-p^{-s})^{-2} \prod_{p\equiv 3 \bmod 4} (1-p^{-2s})^{-1} \\
&= (1-2^{-s})^{-1} \prod_{p\equiv 1 \bmod 4} (1-p^{-s})^{-2} \prod_{p\equiv 3 \bmod 4} (1-p^{-s})^{-1}(1+p^{-s})^{-1} \\
&= L(1,s) \prod_{p\equiv 1 \bmod 4} (1-p^{-s})^{-1} \prod_{p\equiv 3 \bmod 4} (1+p^{-s})^{-1} \\
&= L(1,s)L(\chi,s).
\end{aligned}$$

2.2 Regulators

Let $F/\mathbb{Q}$ be a number field. Let $r_1 = r_1(F)$ be the number of real embeddings of F, and $r_2 = r_2(F)$ the number of conjugate pairs of complex embeddings. Denote its discriminant by d_F.

Say that a set of units in $\mathcal{O}_F$ is *independent* if the subgroup of $\mathcal{O}_F^*$ it generates is free abelian, with the chosen units as generators. Let $r = \operatorname{rank}_{\mathbb{Z}} \mathcal{O}_F^* = r_1 + r_2 - 1$, and choose embeddings $\sigma_1, \ldots, \sigma_{r+1} : F \to \mathbb{C}$ corresponding to the archimedean places of F (including one of each conjugate pair of complex embeddings). Define the *regulator* of a set $\alpha_1, \ldots, \alpha_r$ of units to be

$$R_F(\alpha_i) = \left| \det \begin{pmatrix} c_1 \log|\sigma_1(\alpha_1)| & \cdots & c_1 \log|\sigma_1(\alpha_r)| \\ \vdots & & \vdots \\ c_r \log|\sigma_r(\alpha_1)| & \cdots & c_r \log|\sigma_r(\alpha_r)| \end{pmatrix} \right|$$

where

$$c_i = \begin{cases} 1 & \text{if } \sigma_i \text{ is real,} \\ 2 & \text{if } \sigma_i \text{ is complex.} \end{cases}$$

Note that we omit σ_{r+1} from the definition of regulator. The choice of embedding to omit does not affect the result, because for any $\alpha \in \mathcal{O}_F^*$ we have

$$\sum_{i=1}^{r+1} c_i \log|\sigma_i(\alpha)| = \log \prod_{i=1}^{r+1} |\sigma_i(\alpha)|^{c_i} = 0,$$

so the omitted row with entries $c_{r+1} \log|\sigma_{r+1}(\alpha_j)|$ is (minus) the sum of the rows of the matrix.

The regulators of different sets of units have the following relation.

Lemma 2.2. *Suppose*

$$B = \mu \cdot \langle \beta_1, \ldots, \beta_r \rangle \subset A = \mu(F) \cdot \langle \alpha_1, \ldots, \alpha_r \rangle$$

for β_i and α_i independent sets of units, and $r = \operatorname{rank}_{\mathbb{Z}} \mathcal{O}_F^*$. Then

$$\frac{R_F(\beta_i)}{R_F(\alpha_i)} = [A : B].$$

Thus independent sets of units generating the same subgroup of $\mathcal{O}_F^*/\mu(F)$ have the same regulator. We define the *regulator R_F of F* to be the regulator $R_F(\alpha_i)$ of a set of units $\alpha_1, \ldots, \alpha_r$ with

$$\mathcal{O}_F^* = \mu(F) \cdot \langle \alpha_1, \ldots, \alpha_r \rangle.$$

2.3 Class Number Formula

Theorem 2.3. *The ζ-series of a number field F has a meromorphic continuation (which we also denote ζ_F) to the whole complex plane, with the only pole a simple pole at $s = 1$. Setting*

$$\Lambda_F(s) = \left(2^{-r_2}\pi^{-[F:\mathbb{Q}]}|d_F|^{1/2}\right)^s \Gamma\left(\tfrac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}\zeta_F(s),$$

we have the functional equation

$$\Lambda_F(s) = \Lambda_F(1-s).$$

The ζ-function of F encodes much of the arithmetic data of F in its pole at $s = 1$. Let $w_F = \#\mu(F)$ be the number of roots of unity in F.

Theorem 2.4. *For a number field F, the ζ-function ζ_F has a simple pole at $s = 1$ with residue*

$$\operatorname{res}_{s=1}\zeta_F(s) = 2^{r_1}(2\pi)^{r_2}\frac{h_F R_F}{w_F|d_F|^{1/2}}.$$

Since the Riemann ζ-function (i.e. the Dirichlet L-fuction of the trivial character) has a simple pole at $s = 1$ with residue 1, we can combine Theorem 2.4 with Proposition 2.1 to obtain the analytic class number formula.

Theorem 2.5 (Analytic class number formula)**.** *Let F be a number field. Then*

$$\prod_{\substack{\chi\in X(F)\\ \chi\neq 1}} L(\chi,1) = 2^{r_1}(2\pi)^{r_2}\frac{h_F R_F}{w_F|d_F|^{1/2}}.$$